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THE PROBLEM OF THE SIMPLE SMOOTH CRACK IN AN INFINITE ANISOTROPIC ELASTIC MEDIUM

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Abstract—The problem of the simple smooth curvilinear crack in an infinite anisotropic elastic medium under conditions of generalized plane stress or plane strain and under the supposition that the plane of the problem is a plane of elastic symmetry of the anisotropic medium is reduced to a complex Cauchy-type singular integral equation along the crack together with a condition of single-valuedness of displacements around the crack by using the complex potentials technique. Application to the case of a straight crack is also given.

1. INTRODUCTION

The problem of the simple smooth crack in an infinite anisotropic elastic medium under conditions of generalized plane stress or plane strain and under the normally made supposition that the plane of the problem is a plane of elastic symmetry of the anisotropic medium seems not to have been solved in its general form although the corresponding problem for an isotropic medium has been completely solved[1–4]. Only the case of a straight crack in an infinite anisotropic elastic medium has been studied, due to its simplicity.

Thus, Savin[5], Milne–Thomson[6] and Galidakis[7] have considered special cases of the first fundamental problem for a simple straight crack in an infinite anisotropic medium by using the method of conformal mapping of the crack on the unit circle or, in another way, by considering the crack as an extreme case of an ellipse in an infinite anisotropic medium. Particularly, the stress field near the crack tip has been studied by Sih, Paris and Irwin[8].

One easier way of treating the problem of a straight crack in an infinite anisotropic medium is by reducing it to a Riemann-Hilbert problem along the crack through the use of the method of complex potentials in a way analogous to that used by Muskhelishvili[9] for the case of an isotropic medium. This method has the advantages over the above-mentioned method of conformal mapping that it is relatively simpler and can be applied to the case of multiple collinear cracks. By this approach, the first fundamental problem for collinear cracks in an infinite anisotropic medium was treated by Green and Zerna[10] in one special case. More general cases of the same problem have been considered by Sih and Liebowitz[11] and Ioakimidis[12], while the most general case has been recently studied by Krenk[13].

A third way of studying the problem of a simple straight crack in an infinite anistoropic medium is by considering the crack as composed of a series of elementary and infinitely close to each other dislocations. This approach was used by Barnett and Asaro[14] and Tupholme[15], while Stroh[16] has found the stress fields caused by dislocations and cracks in plane anisotropic media. The main advantage of this method consists in the fact that it is applicable not only to the case when the plane of the problem is a plane of elastic symmetry, but also to the case when this plane has an arbitrary orientation, when the method of complex potentials does not work.

For the solution of the first fundamental problem of a simple smooth curvilinear crack in an infinite anisotropic elastic medium, both the method of conformal mapping and the method of reducing this problem to a Riemann-Hilbert boundary value problem cannot be easily used. Only the method of dislocations may be used in the case when the applied external stresses on the two edges of the crack have the same distribution. In the opposite case, this method should be complemented by considering not only dislocations but also concentrated forces along the crack. This method will not be used here, but it is easy to prove that it is equivalent to the method used here.

In this paper, the method of complex potentials for the anisotropic plane elasticity, developed by Lekhnitskii[17] and others, will be used, together with the Plemelj formulae, for the reduction of the problem under consideration to a complex Cauchy-type singular integral equation along the crack in conjuction with the condition of single-valuedness of displacements. This integral equation can be easily reduced to a pair of Cauchy-type real singular integral equations, which can be further solved by conversion to a system of linear equations and application of either the Gauss–Chebyshev method, developed by Erdogan and Gupta[18], or the Lobatto–Chebyshev method, developed by Theocaris and Ioakimidis[19].

2. GENERAL FORMULAE

Consider a simple smooth crack L in an infinite anisotropic medium. The crack is loaded on both its edges (+) and (−) by normal and shear loading, $\sigma_n^{\pm}(t)$ and $\sigma_t^{\pm}(t)$ respectively. Also the stresses at infinity $\sigma_{xx\infty}$, $\sigma_{yy\infty}$ and $\tau_{xy\infty}$ are considered to be known. We will try to find the complex potentials $\Phi_0(z_1)$ and $\Psi_0(z_2)$ of the theory of plane elasticity for anisotropic media[17], where variables z_1 and z_2 are related to the Cartesian coordinates x and y through relations

$$z_1 = x + \mu_1 y, \qquad z_2 = x + \mu_2 y, \tag{1}$$

where μ_1 and μ_2 are complex constants characterizing the material of the anisotropic medium under consideration.

The stress components σ_{xx}, σ_{yy} and τ_{xy} may be found at any point of the anisotropic plane through the complex potentials $\Phi_0(z_1)$ and $\Psi_0(z_2)$ by using the following formulae[5]

$$\begin{aligned}\sigma_{xx} &= 2Re\,\{\mu_1^2\Phi_0(z_1) + \mu_2^2\Psi_0(z_2)\},\\ \sigma_{yy} &= 2Re\,\{\Phi_0(z_1) + \Psi_0(z_2)\},\\ \tau_{xy} &= -2Re\,\{\mu_1\Phi_0(z_1) + \mu_2\Psi_0(z_2)\}.\end{aligned} \tag{2}$$

As regards the normal and shear components of stresses, σ_n and σ_t respectively, on the two edges of the crack, they can be determined by using the following relation[9]

$$\sigma_n + i\sigma_t = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) - \frac{e^{-2i\vartheta}}{2}(\sigma_{xx} - \sigma_{yy} + 2i\tau_{xy}), \tag{3}$$

where ϑ is the angle subtended by the tangent of the crack at a point t of the crack and the Ox axis. This relation is valid for both isotropic and anisotropic media, while relations (2) are valid only for anisotropic media.

Introducing expressions (2) of stress components σ_{xx}, σ_{yy} and τ_{xy} in relations (3), we obtain

$$\begin{aligned}\sigma_n + i\sigma_t = {} & Re\,\{(1+\mu_1^2)\Phi_0(t_1) + (1+\mu_2^2)\Psi_0(t_2)\}\\ & + \frac{\overline{\mathrm{d}t}}{\mathrm{d}t}\{Re\,\{(1-\mu_1^2)\Phi_0(t_1) + (1-\mu_2^2)\Psi_0(t_2)\}\\ & + 2iRe\,\{\mu_1\Phi_0(t_1) + \mu_2\Psi_0(t_2)\}\},\end{aligned} \tag{4}$$

after taking into account that

$$\frac{\mathrm{d}t}{\overline{\mathrm{d}t}} = e^{2i\vartheta}, \tag{5}$$

where t are the points of the crack L of Fig. 1, while t_1 and t_2 are the points resulting from points t according to relations (1).

If the point t moves along the crack L, the points t_1 and t_2 move on two representations L_1 and L_2 of this crack determined by relations (1) (Fig. 1). If α and β are the tips of the crack L, then α_1, β_1 and α_2, β_2 are the tips of the arcs L_1 and L_2 respectively.

One can note that eqn (4) can result in another way as follows: We take into account that the components X_n and Y_n of the loading of the crack along the axes Ox and Oy respectively are related to the complex potentials $\varphi_0(z_1)$ and $\psi_0(z_2)$ along the arcs L_1 and L_2 respectively through

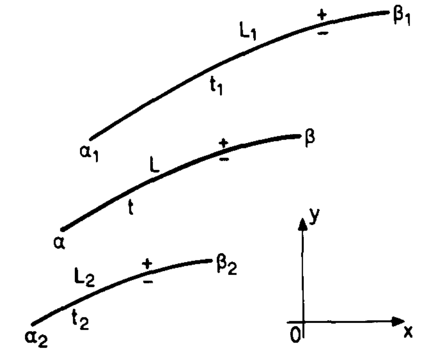

Fig. 1. A simple smooth crack in an infinite anisotropic medium.

relations

$$\varphi_0(z_1)+\overline{\varphi_0(z_1)}+\psi_0(z_2)+\overline{\psi_0(z_2)}=-\int_0^s Y_n\,\mathrm{d}s+C_1,$$

$$\mu_1\varphi_0(z_1)+\bar{\mu}_1\overline{\varphi_0(z_1)}+\mu_2\psi_0(z_2)+\bar{\mu}_2\overline{\psi_0(z_2)}=\int_0^s X_n\,\mathrm{d}s+C_2, \tag{6}$$

where s is the crack length along the crack L and the complex potentials $\varphi_0(z_1)$ and $\psi_0(z_2)$ are related to the above-mentioned complex potentials $\Phi_0(z_1)$ and $\Psi_0(z_2)$ by

$$\Phi_0(z_1)=\varphi_0'(z_1),\quad \Psi_0(z_2)=\psi_0'(z_2). \tag{7}$$

From eqns (6) it can be further deduced that

$$\int_0^s(\sigma_n+i\sigma_t)\,\mathrm{d}\tau=i\int_0^s(X_n+iY_n)\,\mathrm{d}s=(1+i\mu_1)\varphi_0(t_1)$$

$$+(1+i\bar{\mu}_1)\overline{\varphi_0(t_1)}+(1+i\mu_2)\psi_0(t_2)+(1+i\bar{\mu}_2)\overline{\psi_0(t_2)}+C_1+iC_2. \tag{8}$$

Because of relations (1), we can find that the points t_1 and t_2 of curves L_1 and L_2 are related to the corresponding points t of crack L by

$$t_1=\frac{1}{2}\{(1-i\mu_1)t+(1+i\mu_1)\bar{t}\},\quad t_2=\frac{1}{2}\{(1-i\mu_2)t+(1+i\mu_2)\bar{t}\}. \tag{9}$$

Equation (9) differentiated with respect to t yields

$$\frac{\mathrm{d}t_1}{\mathrm{d}t}=\frac{1}{2}\left\{(1-i\mu_1)+(1+i\mu_1)\frac{\overline{\mathrm{d}t}}{\mathrm{d}t}\right\},\quad \frac{\mathrm{d}t_2}{\mathrm{d}t}=\frac{1}{2}\left\{(1-i\mu_2)+(1+i\mu_2)\frac{\overline{\mathrm{d}t}}{\mathrm{d}t}\right\},$$

$$\frac{\mathrm{d}t_1}{\overline{\mathrm{d}t}}=\frac{1}{2}\left\{(1-i\mu_1)\frac{\mathrm{d}t}{\overline{\mathrm{d}t}}+(1+i\mu_1)\right\},\quad \frac{\mathrm{d}t_2}{\overline{\mathrm{d}t}}=\frac{1}{2}\left\{(1-i\mu_2)\frac{\mathrm{d}t}{\overline{\mathrm{d}t}}+(1+i\mu_2)\right\}. \tag{10}$$

Now, eqn (8) differentiated with respect to t and because of relations (10) yields relation (4).

As regards the loading at infinity, which is asumed to be constant, we consider that functions $\Phi_0(z_1)$ and $\Psi_0(z_2)$ tend to definite values Γ and Γ' for $|z_{1,2}|\to\infty$, having thus the forms

$$\Phi_0(z_1)=\Gamma+\Phi(z_1),$$

$$\Psi_0(z_2)=\Gamma'+\Psi(z_2), \tag{11}$$

where the functions $\Phi(z_1)$ and $\Psi(z_2)$ tend to zero for $z_{1,2}\to\infty$, that is

$$\Phi(z_1)=0\left(\frac{1}{z_1}\right),\quad \Psi(z_2)=0\left(\frac{1}{z_2}\right),\quad \text{for}\quad |z_{1,2}|\to\infty. \tag{12}$$

The constants Γ and Γ' can be easily determined from values $\sigma_{xx\infty}$, $\sigma_{yy\infty}$ and $\tau_{xy\infty}$ of the stresses at infinity through the system of linear equations

$$\begin{aligned}
&\Gamma - \bar{\Gamma} = 0, \\
&\mu_1{}^2\Gamma + \bar{\mu}_1{}^2\bar{\Gamma} + \mu_2{}^2\Gamma' + \bar{\mu}_2{}^2\bar{\Gamma}' = \sigma_{xx\infty}, \\
&\Gamma + \bar{\Gamma} + \Gamma' + \bar{\Gamma}' = \sigma_{yy\infty}, \\
&\mu_1\Gamma + \bar{\mu}_1\bar{\Gamma} + \mu_2\Gamma' + \bar{\mu}_2\bar{\Gamma}' = -\tau_{xy\infty},
\end{aligned} \tag{13}$$

the first of which is arbitrary and was used instead of the condition of rotation at infinity [5], while the next three equations result from eqns (2) together with expressions (11) of complex potentials $\Phi_0(z_1)$ and $\Psi_0(z_2)$.

By considering now the constants Γ and Γ' as known, we have to determine the complex functions $\Phi(z_1)$ and $\Psi(z_2)$, instead of $\Phi_0(z_1)$ and $\Psi_0(z_2)$ respectively.

3. REDUCTION OF THE PROBLEM TO A SINGULAR INTEGRAL EQUATION

For the reduction of the problem under consideration to a complex Cauchy-type singular integral equation, we must use the boundary condition (4), which must be fulfilled on both edges of the crack L. By inserting into this condition expressions (11) for the complex potentials $\Phi_0(z_1)$ and $\Psi_0(z_2)$ we obtain

$$\begin{aligned}
&(1+\mu_1{}^2)\Phi(t_1) + (1+\bar{\mu}_1{}^2)\overline{\Phi(t_1)} + (1+\mu_2{}^2)\Psi(t_2) + (1+\bar{\mu}_2{}^2)\overline{\Psi(t_2)} \\
&\quad + \frac{\overline{\mathrm{d}t}}{\mathrm{d}t}\left\{(1+i\mu_1)^2\Phi(t_1) + (1+i\bar{\mu}_1)^2\overline{\Phi(t_1)} + (1+i\mu_2)^2\Psi(t_2) + (1+i\bar{\mu}_2)^2\overline{\Psi(t_2)}\right\} = 2f(t),
\end{aligned} \tag{14}$$

where the function $f(t)$ is defined as

$$\begin{aligned}
f(t) = \sigma_n(t) + i\sigma_t(t) &- \frac{1}{2}\{(1+\mu_1{}^2)\Gamma + (1+\bar{\mu}_1{}^2)\bar{\Gamma} + (1+\mu_2{}^2)\Gamma' + (1+\bar{\mu}_2{}^2)\bar{\Gamma}'\} \\
&- \frac{1}{2}\frac{\overline{\mathrm{d}t}}{\mathrm{d}t}\{(1+i\mu_1)^2\Gamma + (1+i\bar{\mu}_1)^2\bar{\Gamma} + (1+i\mu_2)^2\Gamma' + (1-i\bar{\mu}_2)^2\bar{\Gamma}'\}.
\end{aligned} \tag{15}$$

Equation (14) can be easily written under the form

$$\begin{aligned}
&(1-i\mu_1)\left\{(1-i\mu_1) + \frac{\overline{\mathrm{d}t}}{\mathrm{d}t}(1+i\mu_1)\right\}\Phi(t_1) + (1-i\bar{\mu}_1) \\
&\quad\times\left\{(1-i\bar{\mu}_1) + \frac{\overline{\mathrm{d}t}}{\mathrm{d}t}(1+i\bar{\mu}_1)\right\}\overline{\Phi(t_1)} + (1-i\mu_2)\left\{(1-i\mu_2) + \frac{\overline{\mathrm{d}t}}{\mathrm{d}t}(1+i\mu_2)\right\} \\
&\quad\times\Psi(t_2) + (1-i\bar{\mu}_2)\left\{(1-i\bar{\mu}_2) + \frac{\overline{\mathrm{d}t}}{\mathrm{d}t}(1+i\bar{\mu}_2)\right\}\overline{\Psi(t_2)} = 2\frac{\overline{\mathrm{d}t}}{\mathrm{d}t}\overline{f(t)}.
\end{aligned} \tag{16}$$

Equations (14) and (16) have the disadvantage that four unknown complex functions, $\Phi(z_1)$, $\bar{\Phi}(z_1)$, $\Psi(z_2)$ and $\bar{\Psi}(z_2)$, appear in them. But, as it will be shown in the sequel, it would be advantageous if one of these functions could be deleted. To achieve this deletion, we take into account that the above functions are complex conjugate by pairs, when from eqn (16) as well as its complex conjugate we find that

$$\begin{aligned}
&(\mu_1-\bar{\mu}_2)\left\{(1-i\mu_1) + \frac{\overline{\mathrm{d}t}}{\mathrm{d}t}(1+i\mu_1)\right\}\Phi(t_1) + (\bar{\mu}_1-\bar{\mu}_2)\left\{(1-i\bar{\mu}_1) + \frac{\overline{\mathrm{d}t}}{\mathrm{d}t}(1+i\bar{\mu}_1)\right\} \\
&\quad\times\overline{\Phi(t_1)} + (\mu_2-\bar{\mu}_2)\left\{(1-i\mu_2) + \frac{\overline{\mathrm{d}t}}{\mathrm{d}t}(1+i\mu_2)\right\}\Psi(t_2) = g(t),
\end{aligned} \tag{17}$$

with the function $\bar{\Psi}(z_2)$ deleted and the function $g(t)$ defined as

$$g(t) = -i(1 - i\bar{\mu}_2)f(t) + i\frac{\overline{\mathrm{d}t}}{\mathrm{d}t}(1 + i\bar{\mu}_2)\overline{f(t)}. \tag{18}$$

In another way, we can use relation (8) and its complex conjugate and delete the complex potential $\bar{\psi}_0(z)$ when we have

$$\begin{aligned}(\mu_1 - \bar{\mu}_2)\varphi_0(t_1) + (\bar{\mu}_1 - \bar{\mu}_2)\overline{\varphi_0(t_1)} + (\mu_2 - \bar{\mu}_2)\psi_0(t_2)\\ = \frac{1}{2i}\left\{(1 - i\bar{\mu}_2)\int_0^s (\sigma_n + i\sigma_t)\,\mathrm{d}\tau - (1 + i\bar{\mu}_2)\int_0^s (\sigma_n - i\sigma_t)\,\overline{\mathrm{d}\tau}\right\}.\end{aligned} \tag{19}$$

Differentiating eqn (19) with respect to t and because of relations (7), (11) and (15), eqn (17) is obtained. This equation must be valid on both edges of the crack L, denoted by the symbols (+) and (−).

Since functions $\Phi(z_1)$ and $\Psi(z_2)$ are sectionally holomorphic functions in the whole plane except the arcs L_1 and L_2 respectively, without poles either in the finite part of the complex plane or at infinity, they can be expressed, because of relations (12), through Cauchy-type integrals with densities $\varphi(t_1)$ and $y(t_2)$ as follows

$$\Phi(z_1) = \frac{1}{2\pi i}\int_{L_1} \frac{\varphi(\tau_1)}{\tau_1 - z_1}\,\mathrm{d}\tau_1, \quad \Psi(z_2) = \frac{1}{2\pi i}\int_{L_2} \frac{y(\tau_2)}{\tau_2 - z_2}\,\mathrm{d}\tau_2. \tag{20}$$

It must be noted that subscripts ($_1$) and ($_2$) below the complex variables τ and z have no real meaning and could be omitted, but they are used to remind us that functions $\Phi(z)$ and $\Psi(z)$ appear in the problem under consideration with the variable z replaced by variables z_1 and z_2 respectively. In this way, no confusion may arise.

Further, because of Plemelj's formulae, we obtain for the boundary values of the function $\Phi(z)$ on the arc L_1

$$\begin{aligned}\Phi^+(t_1) - \Phi^-(t_1) &= \varphi(t_1),\\ \Phi^+(t_1) + \Phi^-(t_1) &= \frac{1}{\pi i}\int_{L_1} \frac{\varphi(\tau)}{\tau - t_1}\,\mathrm{d}\tau\end{aligned} \tag{21}$$

and in a similar way for the boundary values of the function $\Psi(z)$ on the arc L_2

$$\begin{aligned}\Psi^+(t_2) - \Psi^-(t_2) &= y(t_2),\\ \Psi^+(t_2) + \Psi^-(t_2) &= \frac{1}{\pi i}\int_{L_2} \frac{y(\tau)}{\tau - t_2}\,\mathrm{d}\tau.\end{aligned} \tag{22}$$

Now, relation (17), written for both edges of the crack L, takes the form

$$\begin{aligned}(\mu_1 - \bar{\mu}_2)\left\{(1 - i\mu_1) + \frac{\overline{\mathrm{d}t}}{\mathrm{d}t}(1 + i\mu_1)\right\}\Phi^{\pm}(t_1) + (\bar{\mu}_1 - \bar{\mu}_2)\\ \times\left\{(1 - i\bar{\mu}_1) + \frac{\overline{\mathrm{d}t}}{\mathrm{d}t}(1 + i\bar{\mu}_1)\right\}\overline{\Phi^{\pm}(t_1)} + (\mu_2 - \bar{\mu}_2)\\ \times\left\{(1 - i\mu_2) + \frac{\overline{\mathrm{d}t}}{\mathrm{d}t}(1 + i\mu_2)\right\}\Psi^{\pm}(t_2) = g^{\pm}(t).\end{aligned} \tag{23}$$

By addition and subtraction of eqns (23), we obtain

$$
\begin{aligned}
(\mu_1-\bar{\mu}_2)&\left\{(1-i\mu_1)+\frac{\overline{dt}}{dt}(1+i\mu_1)\right\}\frac{1}{\pi i}\int_{L_1}\frac{\varphi(\tau_1)}{\tau_1-t_1}d\tau_1 \\
&-(\bar{\mu}_1-\bar{\mu}_2)\left\{(1-i\bar{\mu}_1)+\frac{\overline{dt}}{dt}(1+i\bar{\mu}_1)\right\}\frac{1}{\pi i}\int_{L_1}\frac{\overline{\varphi(\tau_1)}}{\bar{\tau}_1-\bar{t}_1}\overline{d\tau_1} \\
&+(\mu_2-\bar{\mu}_2)\left\{(1-i\mu_2)+\frac{\overline{dt}}{dt}(1+i\mu_2)\right\}\frac{1}{\pi i}\int_{L_2}\frac{y(\tau_2)}{\tau_2-t_2}d\tau_2=g^+(t)+g^-(t),
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
(\mu_1-\bar{\mu}_2)&\left\{(1-i\mu_1)+\frac{\overline{dt}}{dt}(1+i\mu_1)\right\}\varphi(t_1)+(\bar{\mu}_1-\bar{\mu}_2)\left\{(1-i\bar{\mu}_1)+\frac{\overline{dt}}{dt}(1+i\bar{\mu}_1)\right\}\overline{\varphi(t_1)}+(\mu_2-\bar{\mu}_2) \\
&\times\left\{(1-i\mu_2)+\frac{\overline{dt}}{dt}(1+i\mu_2)\right\}y(t_2)=g^+(t)-g^-(t),
\end{aligned}
$$

where relations (21) and (22) were also taken into account. Equations (24) can be written in a simpler way, because of eqns (10), as follows

$$
\begin{aligned}
&\frac{\mu_1-\bar{\mu}_2}{\pi i}\frac{dt_1}{dt}\int_{L_1}\frac{\varphi(\tau_1)}{\tau_1-t_1}d\tau_1-\frac{\bar{\mu}_1-\bar{\mu}_2}{\pi i}\frac{\overline{dt_1}}{dt}\int_{L_1}\frac{\overline{\varphi(\tau_1)}}{\bar{\tau}_1-\bar{t}_1}\overline{d\tau_1} \\
&\qquad+\frac{\mu_2-\bar{\mu}_2}{\pi i}\frac{dt_2}{dt}\int_{L_2}\frac{y(\tau_2)}{\tau_2-t_2}d\tau_2=p(t), \\
&(\mu_1-\bar{\mu}_2)\frac{dt_1}{dt}\varphi(t_1)+(\bar{\mu}_1-\bar{\mu}_2)\frac{\overline{dt_1}}{dt}\overline{\varphi(t_1)}+(\mu_2-\bar{\mu}_2)\frac{dt_2}{dt}y(t_2)=q(t),
\end{aligned}
\tag{25}
$$

where the functions $p(t)$ and $q(t)$ are defined on the crack L as

$$
p(t)=\frac{1}{2}[g^+(t)+g^-(t)],\quad q(t)=\frac{1}{2}[g^+(t)-g^-(t)].
\tag{26}
$$

In eqns (25), functions $\varphi(t_1)$ and $y(t_2)$, which are representing the densities of Cauchy integrals (20), are unknown and must be determined, in order that the functions $\Phi(z_1)$ and $\Psi(z_2)$ be determined too. For the solution of the system of eqns (25), we can express the unknown function $y(t_2)$ through the unknown function $\varphi(t_1)$ as

$$
y(t_2)=\frac{1}{\mu_2-\bar{\mu}_2}\frac{dt}{dt_2}q(t)-\frac{\mu_1-\bar{\mu}_2}{\mu_2-\bar{\mu}_2}\frac{dt_1}{dt_2}\varphi(t_1)-\frac{\bar{\mu}_1-\bar{\mu}_2}{\mu_2-\bar{\mu}_2}\frac{\overline{dt_1}}{dt_2}\overline{\varphi(t_1)},
\tag{27}
$$

because of the second of eqns (25). Introducing now the function $y(t_2)$, given by the expression (27), in the first of eqns (25), we obtain a Cauchy-type singular integral equation for the determination of the function $\varphi(t_1)$, which is expressed as

$$
\begin{aligned}
&\frac{\mu_1-\bar{\mu}_2}{\pi i}\int_{L_1}\left\{\frac{1}{\tau_1-t_1}\frac{dt_1}{dt}-\frac{1}{\tau_2-t_2}\frac{dt_2}{dt}\right\}\varphi(\tau_1)d\tau_1 \\
&\quad-\frac{\bar{\mu}_1-\bar{\mu}_2}{\pi i}\cdot\int_{L_1}\left\{\frac{1}{\bar{\tau}_1-\bar{t}_1}\frac{\overline{dt_1}}{dt}+\frac{1}{\tau_2-t_2}\frac{dt_2}{dt}\right\}\overline{\varphi(\tau_1)}\overline{d\tau_1}=p(t)-\frac{dt_2}{dt}\frac{1}{\pi i}\int_L\frac{q(\tau)}{\tau_2-t_2}d\tau.
\end{aligned}
\tag{28}
$$

After the determination of the function $\varphi(t_1)$, the complex function $\Phi(z_1)$ will be determined from the first of eqns (20), while the complex function $\Psi(z_2)$ will be determined from the following equation

$$
\begin{aligned}
\Psi(z_2)=&\frac{1}{2\pi i(\mu_2-\bar{\mu}_2)}\int_L\frac{q(\tau)}{\tau_2-z_2}d\tau-\frac{\mu_1-\bar{\mu}_2}{2\pi i(\mu_2-\bar{\mu}_2)}\int_{L_1}\frac{\varphi(\tau_1)}{\tau_2-z_2}d\tau_1 \\
&-\frac{\bar{\mu}_1-\bar{\mu}_2}{2\pi i(\mu_2-\bar{\mu}_2)}\int_{L_1}\frac{\overline{\varphi(\tau_1)}}{\tau_2-z_2}\overline{d\tau_1}
\end{aligned}
\tag{29}
$$

resulting from the second of eqns (20) if eqn (27) is also taken into account.

4. THE CONDITION OF SINGLE-VALUEDNESS OF DISPLACEMENTS

The unknown function $\varphi(t_1)$ in the singular integral eqn (28) must also satisfy the condition of single-valuedness of displacements $u(z)$ and $v(z)$ around crack L. We can obtain this condition by taking into consideration the following formulae[5]

$$u(z) = 2Re\,\{p_1\varphi_0(z_1) + p_2\psi_0(z_2)\},$$
$$v(z) = 2Re\,\{q_1\varphi_0(z_1) + q_2\psi_0(z_2)\}, \tag{30}$$

where the constants p_1, p_2, q_1 and q_2 are characteristic quantities of the material of the anisotropic medium.

The condition of single-valuedness of displacements around the crack L may be written as

$$\int_L \left\{\frac{d[u^+(\tau) - u^-(\tau)]}{d\tau} + i\frac{d[v^+(\tau) - v^-(\tau)]}{d\tau}\right\} d\tau = 0, \tag{31}$$

where the symbols $u^\pm(t)$ and $v^\pm(t)$ denote the values of displacements on the two edges of the crack. This condition can be also written under the form

$$(p_1 + iq_1)\int_{L_1} [\Phi^+(\tau_1) - \Phi^-(\tau_1)]\, d\tau_1 + (\bar{p}_1 + i\bar{q}_1)\int_{L_1} [\overline{\Phi^+(\tau_1)} - \overline{\Phi^-(\tau_1)}]\, \overline{d\tau_1} + (p_2 + iq_2)$$
$$\times \int_{L_2} [\Psi^+(\tau_2) - \Psi^-(\tau_2)]\, d\tau_2 + (\bar{p}_2 + i\bar{q}_2)\int_{L_2} [\overline{\Psi^+(\tau_2)} - \overline{\Psi^-(\tau_2)}]\, \overline{d\tau_2} = 0, \tag{32}$$

where the following relation, resulting by a differentiation of relation (30), was also taken into account

$$\frac{du^\pm(t)}{dt} + i\frac{dv^\pm(t)}{dt} = (p_1 + iq_1)\frac{dt_1}{dt}\Phi^\pm(t_1) + (\bar{p}_1 + i\bar{q}_1)\frac{\overline{dt_1}}{dt}\overline{\Phi^\pm(t_1)}$$
$$+ (p_2 + iq_2)\frac{dt_2}{dt}\Psi^\pm(t_2) + (\bar{p}_2 + i\bar{q}_2)\frac{\overline{dt_2}}{dt}\overline{\Psi^\pm(t_2)}, \tag{33}$$

as well as eqns (11).

Furthermore, the condition (32), because of eqns (21), (22) and (27), may be written as

$$\{(p_1 + iq_1)(\mu_2 - \bar{\mu}_2) - (p_2 + iq_2)(\mu_1 - \bar{\mu}_2) + (\bar{p}_2 + i\bar{q}_2)(\mu_1 - \mu_2)\}\int_{L_1} \varphi(\tau_1)\, d\tau_1$$
$$+ \{(\bar{p}_1 + i\bar{q}_1)(\mu_2 - \bar{\mu}_2) - (p_2 + iq_2)(\bar{\mu}_1 - \bar{\mu}_2) + (\bar{p}_2 + i\bar{q}_2)(\bar{\mu}_1 - \mu_2)\}\int_{L_1} \overline{\varphi(\tau_1)}\, \overline{d\tau_1}$$
$$= -(p_2 + iq_2)\int_L q(\tau)\, d\tau + (\bar{p}_2 + i\bar{q}_2)\int_L \overline{q(\tau)}\, \overline{d\tau}. \tag{34}$$

This equation, equivalent to the condition of single-valuedness of displacements (31), should be verified by the unknown function $\varphi(t_1)$ of the singular integral eqn (28), which, otherwise, could not be fully determined.

5. APPLICATION TO THE CASE OF A STRAIGHT CRACK

The singular integral eqn (28) has not, in general, a closed-form solution. One of the cases, when such a solution can be easily found, is the case of a straight crack in an infinite anisotropic medium. This crack is supposed to be a part L of the real axis, when the variables t, t_1 and t_2 coincide, as can be seen from relations (9).

In this case eqn (28) can be written as

$$\frac{2(\mu_1 - \mu_2)}{\pi i}\int_L \frac{\varphi(\tau)}{\tau - t}\, d\tau = \overline{p(t)} + \frac{1}{\pi i}\int_L \frac{\overline{q(\tau)}}{\tau - t}\, d\tau, \tag{35}$$

and its general solution is [9]

$$2(\mu_1-\mu_2)\varphi(t)=\overline{q(t)}+\frac{1}{\pi i X(t)}\int_L \frac{X(\tau)\overline{p(\tau)}}{\tau-t}\,\mathrm{d}\tau+\frac{2C}{X(t)}, \tag{36}$$

where C is a constant to be determined and the function $X(z)$ is defined as [9]

$$X(z)=(z-\alpha)^{1/2}(z-\beta)^{1/2}, \quad X(t)\equiv X^+(t)=-X^-(t), \tag{37}$$

where α and β are the end-points of the crack L.

As regards the functions $\overline{p(t)}$ and $\overline{q(t)}$ on the crack L, because of eqns (3), (15), (18) and (26), they are expressed by

$$\begin{aligned}\overline{p(t)}&=-\mu_2(\sigma_{yy}^{+}+\sigma_{yy}^{-})-(\tau_{xy}^{+}+\tau_{xy}^{-})+2(\mu_2\sigma_{yy\infty}+\tau_{xy\infty}),\\ \overline{q(t)}&=-\mu_2(\sigma_{yy}^{+}-\sigma_{yy}^{-})-(\tau_{xy}^{+}-\tau_{xy}^{-}).\end{aligned} \tag{38}$$

Furthermore, from eqns (20) and (27), we find for the functions $\Phi(z)$ and $\Psi(z)$

$$\begin{aligned}\Phi(z)=\frac{1}{2(\mu_1-\mu_2)}&\left\{-\frac{1}{2\pi i}\int_L \frac{\mu_2(\sigma_{yy}^{+}-\sigma_{yy}^{-})+(\tau_{xy}^{+}-\tau_{xy}^{-})}{\tau-z}\,\mathrm{d}\tau\right.\\ &-\frac{1}{2\pi i X(z)}\int_L \frac{X(\tau)[\mu_2(\sigma_{yy}^{+}+\sigma_{yy}^{-})+(\tau_{xy}^{+}+\tau_{xy}^{-})]}{\tau-z}\,\mathrm{d}\tau\\ &\left.+(\mu_2\sigma_{yy\infty}+\tau_{xy\infty})\left[1-\frac{2z-(\alpha+\beta)}{2X(z)}\right]+\frac{C}{X(z)}\right\},\\ \Psi(z)=\frac{1}{2(\mu_1-\mu_2)}&\left\{\frac{1}{2\pi i}\int_L \frac{\mu_1(\sigma_{yy}^{+}-\sigma_{yy}^{-})+(\tau_{xy}^{+}-\tau_{xy}^{-})}{\tau-z}\,\mathrm{d}\tau\right.\\ &+\frac{1}{2\pi i X(z)}\int_L \frac{X(\tau)[\mu_1(\sigma_{yy}^{+}+\sigma_{yy}^{-})+(\tau_{xy}^{+}+\tau_{xy}^{-})]}{\tau-z}\,\mathrm{d}\tau\\ &\left.-(\mu_1\sigma_{yy\infty}+\tau_{xy\infty})\left[1-\frac{2z-(\alpha+\beta)}{2X(z)}\right]-\frac{(\mu_1-\bar{\mu}_2)C-(\mu_1-\mu_2)\bar{C}}{(\mu_2-\bar{\mu}_2)X(z)}\right\},\end{aligned} \tag{39}$$

where it was taken into account that

$$\frac{1}{\pi i X(z)}\int_L \frac{X(\tau)}{\tau-z}\,\mathrm{d}\tau=1-\frac{2z-(\alpha+\beta)}{2X(z)}. \tag{40}$$

Finally, the constant C will be determined from the condition (34) of single-valuedness of displacements, which, because of eqns (38) and (39) can be written as

$$\begin{aligned}&\{(p_1+iq_1)(\mu_2-\bar{\mu}_2)-(p_2+iq_2)(\mu_1-\bar{\mu}_2)+(\bar{p}_2+i\bar{q}_2)(\mu_1-\mu_2)\}\frac{1}{2(\mu_1-\mu_2)}\\ &\times\left\{\int_L \overline{q(\tau)}\,\mathrm{d}\tau-2\pi iC\right\}+\{(\bar{p}_1+i\bar{q}_1)(\mu_2-\bar{\mu}_2)-(p_2+iq_2)(\bar{\mu}_1-\bar{\mu}_2)+(\bar{p}_2+i\bar{q}_2)(\bar{\mu}_1-\mu_2)\}\\ &\times\frac{1}{2(\bar{\mu}_1-\bar{\mu}_2)}\left\{\int_L q(\tau)\,\mathrm{d}\tau+2\pi i\bar{C}\right\}=-(p_2+iq_2)\int_L q(\tau)\,\mathrm{d}\tau+(\bar{p}_2+i\bar{q}_2)\int_L \overline{q(\tau)}\,\mathrm{d}\tau.\end{aligned} \tag{41}$$

It can be also remarked that, because of eqn (38), we have

$$\int_L \overline{q(\tau)}\,\mathrm{d}\tau=-\mu_2\int_L(\sigma_{yy}^{+}-\sigma_{yy}^{-})\,\mathrm{d}\tau-\int_L(\tau_{xy}^{+}-\tau_{xy}^{-})\,\mathrm{d}\tau=X+\mu_2Y, \tag{42}$$

where X and Y are the components of the resultant force acting on the whole crack.

The results of this paragraph are in accordance with those found in Ref. [12] by using the method of reduction of the problem of a straight crack to a Riemann–Hilbert boundary value problem and by a closed-form solution of the latter.

6. DISCUSSION—GENERALIZATIONS

The method of treating the problem of a simple smooth curvilinear crack in an infinite anisotropic medium presented in this paper can be easily generalized to a series of other interesting problems for plane anisotropic media which either contain cracks or not. In all cases, the problems are reduced to one complex Cauchy-type singular integral equation along all the boundaries of the media under consideration, accompanied with the necessary conditions of single-valuedness of displacements.

Besides the method presented here the based on the theory of Cauchy-type integrals and the Plemelj formulae, another method for reducing the problem of a finite or infinite anisotropic medium with or without cracks or holes to a complex singular integral equation consists in considering elementary concentrated forces or dislocations acting along its boundaries except cracks and also both concentrated forces and dislocations acting along the cracks.

Also, in an analogous way to that used here for the case of the first fundamental problem, the second fundamental and the mixed fundamental problems could be treated as well, although the resulting singular integral equation for the case of the mixed fundamental problem is a little more complicated as involving the unknown function $\varphi(t)$ not only inside integrals, but also as a free term.

It can be also noted that, instead of the expressions (20) for the functions $\Phi(z_1)$ and $\Psi(z_2)$, we could have used expressions of the form

$$\Phi(z_1)=\frac{1}{2\pi i X_1(z_1)}\int_{L_1}\frac{X_1(\tau_1)\omega(\tau_1)}{\tau_1-z_1}\,d\tau_1+\frac{C_1}{X_1(z_1)},$$

$$\Psi(z_2)=\frac{1}{2\pi i X_2(z_2)}\int_{L_2}\frac{X_2(\tau_2)\chi(\tau_2)}{\tau_2-z_2}\,d\tau_2+\frac{C_2}{X_2(z_2)},\tag{43}$$

where the functions $X_{1,2}(z)$ are given by

$$X_{1,2}(z)=(z-\alpha_{1,2})^{1/2}(z-\beta_{1,2})^{1/2},\tag{44}$$

where $\alpha_{1,2}$ and $\beta_{1,2}$ are the end-points of the arcs L_1 and L_2 of Fig. 1 respectively, and the functions $\omega(\tau_1)$ and $\chi(\tau_2)$ are unknown functions to be determined. This approach was used by Ioakimidis [4], but leads to a more complicated form of a singular integral equation, because of the constants C_1 and C_2 and the functions $X_{1,2}(z)$.

Finally, the numerical solution of the singular integral eqn (28) can be easily found by the Gauss-Chebyshev method proposed by Erdogan and Gupta [18] or by the Lobatto–Chebyshev method proposed by Theocaris and Ioakimidis [19]. In the usual case, when we are interested in the values of the stress intensity factors at the tips of the crack, the Lobatto–Chebyshev method is more accurate and should be preferred.

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